Conventional description of unconventional Coulomb-crystal phase transitions in three-dimensional classical $O(N)$ spin ice

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We study the phase transition between the high-temperature Coulomb phase and the low-temperature staggered crystal phase in three-dimensional classical $O(N)$ spin-ice model. Compared with the previously proposed means CP(1) (complex projective model) formalism on the Coulomb-crystal transition of the classical dimer model, our description based on constrained Ginzburg-Landau formalism is more conventional, due to a fundamental difference between the $O(N)$ spin ice and the dimer model. The situations with cubic symmetry as well as easy-plane and easy-axis anisotropies are all studied. A systematic $\epsilon=4-d$ expansion is used to study the universality class of the phase transitions.

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I. INTRODUCTION

The three-dimensional (3D) classical dimer model (CDM), as the simplest model with an algebraic liquid phase (usually called the Coulomb phase), has attracted many analytical and numerical studies.[1–4] The ensemble of 3D CDM is all the configurations of dimer coverings, which are subject to a local constraint on every site: every site is connected to precisely m dimers with $0<m<6$ (denoted as CDM-m), and most studies are focused on the case with $m=1$. The partition function of the CDM is a summation of all the allowed dimer coverings with a Boltzmann weight that favors certain types of dimer configurations. For instance, the most standard CDM-1 takes the following form:

$$Z=\sum_{\text{dimer configuration}} \exp\left(-\frac{E}{T}\right),$$

$$E=\sum_{\text{plaquettes}} -U(n_{xy}+n_{yz}+n_{zx}). \tag{1}$$

n_{xy}, n_{yz}, and n_{zx} are numbers defined on each unit plaquette in xy, yz, and zx planes, they take value 1 when this plaquette contains two parallel dimers, and take value 0 otherwise. When $U>0$, the ground state of this model favors to have as many parallel dimers as possible, therefore when $T<T_c$, the system develops columnar crystalline dimer patterns in Fig. 1(a) while when $T>T_c$ the system is in the Coulomb phase with power-law correlation between dimer densities, which according to the standard Ginzburg-Landau (GL) theory can only occur at critical points instead of stable phases. This phase diagram has been confirmed by a number of numerical simulations.[1–4]

The nature of the transition at T_c attracts most efforts. If $U>0$, numerical studies confirm that this transition is continuous and the data suggest that at this transition the discrete cubic symmetry is enlarged to an O(3) rotation symmetry.[4] Analytically, to describe this locally constrained dimer system, we can introduce the "magnetic field" $B_{i,\mu}=(n_{i,\mu}-m/6)\eta_i$ with $\mu=+\hat{x},+\hat{y},+\hat{z}$, where $\eta_i=(-1)^i$ is a staggered sign distribution on the cubic lattice. The number $n_{i,\mu}$ is defined on each link (i,μ) between sites i and $i+\mu$, and $n_{i,\mu}=1,0$ represents the presence and absence of dimer. Now the local constraint of the dimer system can be rewritten as a Gauss-law constraint $\vec{\nabla}\cdot\vec{B}=0$. The standard way to solve this Gauss-law constraint is to introduce vector potential $\vec{A}$ defined on the unit plaquettes of the cubic lattice and $\vec{B}=\vec{\nabla}\times\vec{A}$.[5,6] Since vector $\vec{A}$ is no longer subject to any constraint, it is usually assumed that at low energy the system can be described by a local field theory of $\vec{A}$, for instance, the low-energy field theory of the Coulomb phase reads

$$F\sim\int d^3x(\vec{\nabla}\times\vec{A})^2+\cdots, \tag{2}$$

which is invariant under gauge transformation $\vec{A}\to\vec{A}+\vec{\nabla}f$, f is an arbitrary function of space.

Because the dimer number $n_{i,\mu}$ can only take discrete values, the magnetic field $\vec{B}$ and the vector potential $\vec{A}$ are both discrete. Therefore mathematically we should introduce "vertex operator" $L_v\sim\Sigma_\mu\cos[2\pi(A_\mu-\tilde{a}_\mu)]$ to the field theory Eq. (2), and $\tilde{a}_\mu$ is a nonzero background distribution of the vector potential, which is introduced for any nonzero m with $0<m<6$. The Coulomb phase is a phase where this vertex operator is irrelevant perturbatively. The vertex operator will become nonperturbative and drive a phase transition when it

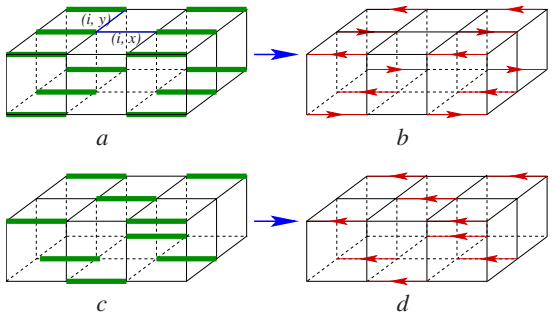

FIG. 1. (Color online) (a) The low-temperature columnar crystalline pattern of Eq. (1) studied in previous references; (b), the magnetic field distribution corresponding to this crystalline pattern (a) with a zero net magnetic field in the large scale; (c), the staggered dimer order; and (d), the net nonzero magnetic field corresponding to the dimer crystal pattern in (c).

is large, and in order to describe this phase transition one can introduce matter fields in the vertex operator which couple to the gauge field minimally

$$L_v \sim \sum_{\mu} \cos[2\pi(\nabla_\mu \theta - A_\mu - \tilde{a}_\mu)] \tag{3}$$

θ is the phase angle of a matter field $\psi \sim e^{i\theta}$. Due to the existence of the nonzero $\tilde{a}_\mu$, ψ moves on a nonzero background magnetic field, the band structure of the matter fields have multiple minima in the Brillouin zone, and the transformation between these minima encodes the information of the lattice symmetry.[7] Therefore in addition to manifesting the discrete nature of the gauge potential A_μ, the condensation of the matter field leads to lattice symmetry breaking, which corresponds to the crystal phase of the CDM. For instance, the transition between the Coulomb and columnar crystal phases of the CDM-1 model is described by the CP(1) model (complex projective model) with an enlarged SU(2) global symmetry.[2,8] This field theory is highly unconventional, in the sense that it is not formulated in terms of physical order parameters. It is expected that more general CDM-m models can also be described by similar Higgs transition although the detailed lattice symmetry transformation for matter fields would depend on m.

One might be tempted to describe the transition between the Coulomb and columnar phases of CDM-1 trough a Ginzburg-Landau approach. One can introduce an O(3) vector $\vec{\varphi}$ with cubic-symmetry anisotropy in favor of six axial directions and $\langle \varphi_\mu \rangle \sim \pm 1$ represents sixfold degenerate columnar order. However, the hedgehog monopole configuration of the O(3) vector $\vec{\varphi}$ always involves a broken dimer, i.e., a defect of the constraint.[7] Then as long as we forbid the presence of the defects, this O(3) model is monopole free and it is well known that the monopole-free O(3) nonlinear sigma model is equivalent to the CP(1) model,[9] which has very different critical exponents from the O(3) Wilson-Fisher fixed point.[10,11]

II. ISOTROPIC $O(N)$ SPIN ICE

In this paper we will study an $O(N)$ generalization of the CDM model. We define an $O(N)$ spin vector S^a with unit length $\Sigma_a (S^a)^2 = 1$ on each link (i,μ) of the cubic lattice, with $a = 1, \ldots, N$, and we assume that the largest term of the Hamiltonian imposes an ice-rule constraint[12] for $O(N)$ spins on six links shared by every site

$$\sum_{\mu=x,y,z} S^a_{i,\mu} + S^a_{i-\mu,\mu} = 0. \tag{4}$$

Systems with this constraint is usually called the spin ice. If $N=1$, each spin can only take value ± 1, therefore every site of the cubic lattice connects to three spins with +1 and three spins with −1, which is equivalent to the CDM-3 model. Just like the CDM, at high temperature, there is a Coulomb phase with algebraic correlation between the $O(N)$ vector S^a while the low-temperature crystal phase is controlled by other interactions. Notice that the $O(N)$ index a and spatial index μ are completely independent from each other. In condensed matter systems, O(3) spins are most common but larger $O(N)$ spin vectors with $N=5$ and 6 can be realized in spin-3/2 ultracold alkali atoms or alkaline earth atoms systems without any fine tuning.[13,14]

In addition to the large constraint Eq. (4), we can design the Hamiltonian as following:

$$E = \sum_{i,\mu,a} J_1 S^a_{i-\mu,\mu} S^a_{i,\mu} + \sum_{i,a} \sum_{\mu \neq \nu} J_2 S^a_{i,\mu} S^a_{i+\nu,\mu}. \tag{5}$$

J_1 is a Heisenberg coupling between spins along the same lattice axis and J_2 is a Heisenberg coupling between spins on two parallel links across a unit square. If $J_1 > 0$ and $J_2 < 0$, in the ground-state spins are antiparallel along the same axis, but parallel between parallel links across a unit square, which is an $O(N)$ analog of the columnar state of the CDM in Fig. 1(a). If J_1 and J_2 are both positive, in the ground state the spins are antiparallel between nearest-neighbor links on the same axis, as well as between parallel links across a unit square, which is an analog of the staggered dimer configuration in Fig. 1(c). In fact, for both cases the zero-temperature ground state of model Eq. (5) has large degeneracy because the spins on x, y, and z axes are ordered independently, and the energy does not depend on the relative angle between $S^a_{i,x}$, $S^a_{i,y}$, and $S^a_{i,z}$, i.e., the ground-state manifold has an enlarged $[O(N)]^3$ symmetry. Compared with the dimer system, the $O(N)$ spin vector is not discrete, therefore although we can still introduce the $O(N)$ vector potential A^a_μ as $\phi^a_\mu = \epsilon_{\mu\nu\rho} \nabla_\nu A^a_\rho$, we cannot introduce the vertex operator as we did for the dimer model, which encodes the information of the discreteness of the dimer variables. Also, it is impossible to write down an $O(N)$ invariant vertex operator. Therefore we need to seek for a different formalism, which is a GL theory of constrained order parameters. The applicability of this formalism to the original CDM-m will be discussed later.

In this paper we will focus on the staggered spin order. Following the magnetic field formalism of the CDM mentioned before, in order to describe this system compactly, we introduce three flavors of $O(N)$ vector field $\phi^a_\mu = S^a_{i,+\mu} \eta_i$ with $\mu = x, y, z$, and now the constraint Eq. (4) can be rewritten concisely as

$$\sum_{\mu} \nabla_\mu \phi^a_\mu = 0. \tag{6}$$

Under the lattice symmetry transformation, ϕ^a_μ transforms as

$$\begin{aligned} &T_\mu: \mu \to \mu + 1, \quad \phi^a_\nu \to -\phi^a_\nu, \quad (\text{for all } \mu, \nu), \\ &R_{\mu,s}: \mu \to -\mu, \quad \phi^a_\mu \to -\phi^a_\mu, \quad \phi^a_\nu \to \phi^a_\nu, \quad (\text{for } \nu \neq \mu), \\ &R_{\mu\nu}: \mu \leftrightarrow \nu, \quad \phi^a_\mu \leftrightarrow \phi^a_\nu, \quad (\text{for } \nu \neq \mu). \end{aligned} \tag{7}$$

T_μ is the translation symmetry along μ direction, $R_{\mu,s}$ is the site centered reflection symmetry, and $R_{\mu\nu}$ is the reflection along a diagonal direction.

The mapping between $S^a_{i,\mu}$ and ϕ^a_μ is very similar to the dimer case in Fig. 1, and the staggered spin order corresponds to the uniform order of ϕ^a_μ, and all flavors of spin vectors are ordered. Therefore presumably ϕ^a_μ are the low-

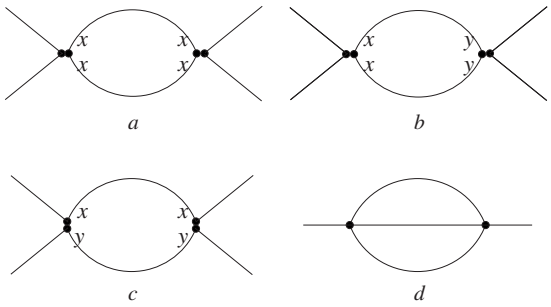

FIG. 2. [(a)–(c)] Momentum-shell loop integrals evaluated to be $A\ln(\Lambda/\tilde{\Lambda})$, $B\ln(\Lambda/\tilde{\Lambda})$, and $C\ln(\Lambda/\tilde{\Lambda})$, respectively, with parameters given by Eq. (12). (d), the self-energy correction that renormalizes γ in Eq. (8), this two-loop diagram will contribute to the renormalization of u, v, and w at order ϵ^3, and hence is negligible in our calculation.

energy modes close to the transition and we can write down the following symmetry allowed trial field theory for ϕ_μ^a with softened unit length constraint:

$$\mathcal{F}=\sum_{\mu,a}\phi_\mu^a(-\nabla^2+r-\gamma\nabla_\mu^2)\phi_\mu^a+\sum_a g\left(\sum_\mu \nabla_\mu\phi_\mu^a\right)^2+\mathcal{F}_4. \tag{8}$$

If we just take the inverse of the Gaussian part of Eq. (8) with $r>0$, we obtain the following correlation function of ϕ_μ^a (with $\gamma=0$ for simplicity):

$$D_{\mu\nu}^{ab}\sim\frac{\delta_{ab}}{r+q^2}\left[\delta_{\mu\nu}-\frac{gq_\mu q_\nu}{r+(1+g)q^2}\right]. \tag{9}$$

In the limit with $g\to\infty$, the constraint Eq. (6) is effectively imposed and the correlation function reads

$$\lim_{g\to\infty}D_{\mu\nu}^{ab}(\vec{q})=\langle\phi_\mu^a(\vec{q})\phi_\nu^b(-\vec{q})\rangle\sim\frac{\delta_{ab}}{r+q^2}P_{\mu\nu},$$

$$P_{\mu\nu}=\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}. \tag{10}$$

$P_{\mu\nu}$ is a projection matrix that projects a vector to the transverse direction, i.e., the limit $g\to\infty$ gives the longitudinal spin wave an infinite energy. After Fourier transformation, this correlation function Eq. (10) gives us the $1/r^3$ power-law spin correlation of the Coulomb phase. When $r<0$, the vector ϕ_μ^a is ordered. We will keep $g\to\infty$ in the renormalization group (RG) calculation, which is equivalent to the ice-rule constraint of the system. Interaction $\mathcal{F}_4$ will not spontaneously generate longitudinal spin wave and we can check this as following: we can first keep g finite at the beginning and calculate the leading order self-energy correction using Fig. 2(d). And in the result after taking the limit $g\to\infty$ the longitudinal wave still acquires infinite kinetic energy and the dressed correlation function is still fully transverse.

In Eq. (8) when $\gamma=0$, the quadratic part of the field theory is invariant under $O(N)\times O(3)$ transformation, the O(3) symmetry is a combined flavor-orbital rotation symmetry. γ term will break this symmetry down to the cubic lattice symmetry and $O(N)$ spin symmetry. The flow of γ comes from the two loop self-energy correction diagram [Fig. 2(d)], and the RG flow of γ will contribute to the RG equation at the order of ϵ^3, which is negligible at the accuracy of our calculation if we take $\epsilon=4-d$ small. Therefore γ is a constant instead of a scaling function in the RG equation, hereafter we will always assume γ is small.

$\mathcal{F}_4$ in Eq. (8) includes all the symmetry allowed quartic terms of ϕ_μ^a

$$\mathcal{F}_4=u\sum_\mu\left[\sum_a(\phi_\mu^a)^2\right]^2+v\sum_{\mu<\nu}\left[\sum_a(\phi_\mu^a)^2\right]\left[\sum_b(\phi_\nu^b)^2\right]+w\sum_{\mu<\nu}\left[\sum_a\phi_\mu^a\phi_\nu^a\right]\left[\sum_b\phi_\mu^b\phi_\nu^b\right]. \tag{11}$$

The u and v terms are invariant under an enlarged symmetry $[O(N)]^3$ while the w term breaks this symmetry down to one single $O(N)$ symmetry plus lattice symmetry. As already mentioned, the ground-state manifold of model Eq. (5) has the same enlarged $[O(N)]^3$ symmetry. However, the w term can be induced with thermal fluctuation through order-by-disorder mechanism[15] or we can simply turn on such extra biquadratic term energetically in the J_1-J_2 model Eq. (5). By adjusting the ratio between u, v, and w in Eq. (11), points with various enlarged symmetry can be found. For instance, if $w+v=2u$, $\mathcal{F}_4$ has a $O(N)\times O(3)$ symmetry, and the O(3) symmetry is the flavor-orbital combined rotation. Just like the J_1-J_2 model on the square lattice,[15] the quadratic coupling $\Sigma_a\phi_\mu^a\phi_\nu^a$ with $\mu\neq\nu$ as well as more complicated quartic terms like $[\Sigma_a\phi_x^a\phi_y^a][\Sigma_b\phi_y^b\phi_z^b]$ break the reflection symmetry of the system, and hence are forbidden.

Now a systematic RG equation can be computed with four parameters u, v, w and r, at the critical point $r=0$ with the correlation function Eq. (10). In the calculation an $\epsilon=4-d$ expansion is used and the accuracy is kept to the first-order ϵ expansion. Based on the spirit of ϵ expansion, all the loop integrals should be evaluated at $d=4$, and because of the flavor-orbital coupling imposed by the constraint Eq. (4), we should generalize our system to four dimension, and also increase the flavor number to $\mu=x,y,z,\tau$. Notice that the flavor-mixing correlation in Eq. (10) significantly increases the number of diagrams that we need to evaluate. Our RG calculation is based on momentum shell integral, since the correlation function Eq. (10) acquires strong momentum direction dependence, after the momentum shell integral the logarithmic correction will depend on the flavor of the loop integrals. For instance, the loop diagrams in Figs. 2(a)–2(c) expanded to the first order of γ is evaluated as

$$A=\frac{\int_{\tilde{\Lambda}<|\vec{q}|<\Lambda}d^4qD_{\mu\mu}(\vec{q})D_{\mu\mu}(-\vec{q})}{\int_{\tilde{\Lambda}<|\vec{q}|<\Lambda}d^4q(1/q^2)^2}=\frac{5}{8}-\frac{7}{40}\gamma+O(\gamma^2),$$

$$B=\frac{\int_{\tilde{\Lambda}<|\vec{q}|<\Lambda}d^4qD_{\mu\nu}(\vec{q})D_{\mu\nu}(-\vec{q})}{\int_{\tilde{\Lambda}<|\vec{q}|<\Lambda}d^4q(1/q^2)^2}=\frac{1}{24}-\frac{1}{40}\gamma+O(\gamma^2),$$

$$C = \frac{\int_{\tilde{\Lambda}<|\vec{q}|<\Lambda} d^4 q D_{\mu\mu}(\vec{q}) D_{\nu\nu}(-\vec{q})}{\int_{\tilde{\Lambda}<|\vec{q}|<\Lambda} d^4 q (1/q^2)^2} = \frac{13}{24} - \frac{23}{120}\gamma + O(\gamma^2). \tag{12}$$

For arbitrary N, the full-coupled RG equation at the first-order ϵ expansion reads

$$\frac{du}{d\ln l} = \epsilon u - 8A(8+N)u^2 - 6N(A+2B)v^2 - 6(A+2B)w^2 - 24(2B+BN)uv - 12(A+2B)vw - 72Buw,$$

$$\frac{dv}{d\ln l} = \epsilon v - 16B(4+N)u^2 - 4(2AN+7BN+4C)v^2 - 4Cw^2 - 16(2A+AN+2BN+4B)uv - 8(2A+9B)vw - 16(A+6B)uw,$$

$$\frac{dw}{d\ln l} = \epsilon w - 64Bu^2 - 16Bv^2 - 4(2A+10B+BN+2C+CN)w^2 - 16(B+2C)vw - 32Auw,$$

$$\frac{dr}{d\ln l} = 2r - 8(2A+AN+6B+3BN)ur - 12(AN+3BN)vr - 12(A+3B)wr. \tag{13}$$

A similar set of recursion relations of quartic interaction terms were computed in a different context in Ref. 16.

Let us first discuss the solution of this RG equation with $\gamma=0$. Solving this equation at $r=0$ with number A,B,C given by Eq. (12), we find eight fixed points, with one stable fixed point for $N \geq N_c = 70$, while for any $N < N_c$ only instable fixed points are found. The analytical expression of the stable fixed point as a function of N with $N > N_c$ can be straightforwardly obtained by solving Eq. (13), but the result is rather lengthy. Instead, we will analyze the solution of Eq. (13) with an expansion of $1/N$. For instance the stable fixed point is located at

$$u_* = \frac{17\epsilon}{84N} - \frac{1139\epsilon}{441N^2} + O\left(\frac{\epsilon}{N^3}\right),$$

$$v_* = -\frac{\epsilon}{42N} - \frac{1964\epsilon}{1323N^2} + O\left(\frac{\epsilon}{N^3}\right),$$

$$w_* = \frac{3\epsilon}{7N} - \frac{4870\epsilon}{1323N^2} + O\left(\frac{\epsilon}{N^3}\right). \tag{14}$$

Since now $v_*+w_*=2u_*$, this fixed point has the enlarged $O(N)\times O(d)$ symmetry mentioned before. Close to the stable fixed point, the three eigenvectors of the RG flow have scaling dimensions

$$\Delta_1 = -\epsilon + O\left(\frac{\epsilon}{N^2}\right),$$

$$\Delta_2 = -\epsilon + \frac{4448\epsilon}{567N} + O\left(\frac{\epsilon}{N^2}\right),$$

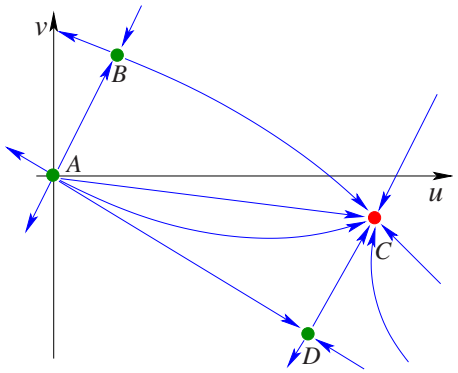

FIG. 3. (Color online) The schematic RG flow diagram in the large-N limit with $w=w_*=\frac{3}{7N}$. The four fixed points are $A,(u_*,v_*)=(0,0)$; $B,(\frac{1}{24N},\frac{1}{12N})$; $C,(\frac{17}{84N},-\frac{1}{42N})$; and $D,(\frac{9}{56N},-\frac{3}{28N})$. C is the stable fixed point.

$$\Delta_3 = -\epsilon + \frac{24950\epsilon}{567N} + O\left(\frac{\epsilon}{N^2}\right). \tag{15}$$

At the stable fixed point Eq. (14), r is the only relevant perturbation, and plugging the fixed point values Eq. (14) back to the RG equation, we obtain the critical scaling dimension

$$[r] = \frac{1}{\nu} = 2 - \epsilon + \frac{158\epsilon}{7N} + O\left(\frac{\epsilon}{N^2}\right). \tag{16}$$

Since at the ground state all three flavors of spin vectors are ordered, in the field theory $\mathcal{F}_4$, v should be smaller than $2u$, which is well consistent with the stable fixed point in Eq. (14) with negative v_*. This fixed point has positive w_*, which favors noncollinear alignment between spins on different axes. Therefore the transition between Coulomb and noncollinear staggered state has a better chance to be described by this fixed point.

Notice that had we included the anisotropic velocity γ into account, its leading RG flow will be at order of ϵ^2, and the flow of γ will contribute to the RG flow of u, v, and w at order of ϵ^3, therefore it is justified to take γ a constant in our calculation as long as we keep ϵ small enough. When γ is nonzero but small, we can solve the RG equation with A, B, and C given by Eq. (12), and the RG flows will only change quantitatively, although the $O(N)\times O(3)$ symmetry of the stable fixed point is broken by γ. Expanded to the first order of γ, the scaling dimensions of the three eigenvectors of the RG equation at the stable fixed point become $\Delta_1=-\epsilon$, $\Delta_2=-\epsilon+\frac{4448\epsilon}{567N}+\frac{2849936\epsilon\gamma}{3988845N}$, $\Delta_3=-\epsilon+\frac{24950\epsilon}{567N}-\frac{30088528\epsilon\gamma}{27921915N}$, and the scaling dimension of r becomes $[r]=\frac{1}{\nu}=2-\epsilon+\frac{158\epsilon}{7N}-\frac{1032\epsilon\gamma}{1715N}$.

If we took the limit $N\to\infty$ in the physical system, we only need to keep the terms linear with N in Eq. (13) and now the equation becomes precise even with $\epsilon=1$. In this case the RG flow of w is decoupled from u and v hence four of the eight fixed points have $w_*=0$, and all the others have $w_*=3/(7N)$. The RG flow diagram for u and v with $w=w_*=3/(7N)$ in the large-N limit is depicted in Fig. 3.

As we promised in the beginning of this paper, we should discuss the applicability of the constrained GL formalism

discussed in this paper to the CDM-3, which corresponds to the case with $N=1$. In our GL formalism, in the ordered phase, the power-law spin-spin correlation still persists if the long-range correlation is subtracted. For instance, the fluctuation $\delta\phi_\mu=\phi_\mu-\langle\phi_\mu\rangle$ is still subject to the constraint $\Sigma_\mu\nabla_\mu\delta\phi_\mu=0$, therefore although the fluctuation is gapped, it still leads to the $1/r^3$ power-law correlation. But in CDM, the crystal phase is more conventional, in the sense that the crystal phase only has short-range connected dimer correlation on top of the long-range order, which can be checked with a low-temperature expansion of CDM.[17] Like what was discussed in the introduction, the key property of the case for $N=1$ is that, the spins only take discrete value ±1, the vertex operator such as Eq. (3) in the dual field theory in terms of vector potential can drive the system to a phase with short-range connected correlation through a Higgs transition. The effect of the vertex operator was missing in our GL formalism.

We can solve Eq. (13) with $N=1$, where v and w terms are identical. In this case in addition to the trivial Gaussian fixed point, there is only one other fixed point at $v_*=2u_*=\epsilon/34$ with O(3) flavor-space combined rotation symmetry, which is the same fixed point as the ferromagnetic transition with dipolar interaction.[18,19] In 3D space, the dipolar interaction also projects a spin wave to its transverse direction. The dipolar fixed point is instable against the O(3) to cubic symmetry breaking, therefore when $N=1$ our first-order ϵ expansion predicts a first-order transition. Based on the discussion in the previous paragraph, one possible scenario for the CDM-3 with staggered ground state is that, if we lower the temperature from the Coulomb phase, after the first-order transition of ϕ^a, there has to be another "Higgs-type" phase transition that destroys the power-law connected correlation. Or there can be one single strong first-order transition that connects the Coulomb phase and staggered dimer crystal directly.

III. $O(N)$ SPIN ICE WITH EASY PLANE ANISOTROPY

Model Eq. (5) is invariant under cubic lattice symmetry and we can certainly turn on various anisotropies to this model, like what was studied in the CDM-1 model.[2] For instance, let us modify the model Eq. (5) slightly

$$E=\sum_{i,\mu,a}J_{1,\mu}S^a_{i-\mu,\mu}S^a_{i,\mu}+\sum_{i,a}\sum_{\mu\neq\nu}J_{2,\mu\nu}S^a_{i,\mu}S^a_{i+\nu,\mu}. \quad (17)$$

If $J_{1,x}=J_{1,y}>J_{1,z}>0$, and $J_{2,xy}=J_{2,xz}=J_{2,yx}=J_{2,yz}>J_{2,zx}=J_{2,zy}>0$, the $O(N)$ spin vectors in the xy plane, $S^a_{i,x}$ and $S^a_{i,y}$ have a stronger tendency to order compared with $S^a_{i,z}$. Therefore when we lower the temperature from the high-temperature algebraic phase, the $O(N)$ vectors in xy plane are expected to order first at critical temperature T_{c1}. Notice that since the linear interflavor coupling is still forbidden by the reflection symmetry in Eq. (7), the order of ϕ^a_x and ϕ^a_y does not imply the order of ϕ^a_z. In the field theory close to T_{c1}, the anisotropy can be described by an extra mass term for $\Sigma_a m^2(\phi^a_z)^2$ in the free energy, which is clearly a relevant perturbation at the critical point $r=0$

$$\mathcal{F}=\sum_{\mu,a}\phi^a_\mu(-\nabla^2+r)\phi^a_\mu+\sum_a m^2(\phi^a_z)^2+\sum_a g\left(\sum_\mu\nabla_\mu\phi^a_\mu\right)^2+\mathcal{F}_4. \quad (18)$$

In the equation above we have taken $\gamma=0$ for simplicity. To calculate the RG equations for $\mathcal{F}_4$, we still need to increase the dimension and flavor number to four with $\mu=x,y,z,\tau$, and the anisotropy of the generalized system will prefer the $O(N)$ vectors to order on three of the four axes. At the critical point $r=0$, due to the relevance of the extra mass term, we can safely take $m\to\infty$, and the correlation function between ϕ^a_μ reads

$$\lim_{g,m\to\infty,r\to0}D^{ab}_{\mu\nu}(\vec{q})=\langle\phi^a_\mu(\vec{q})\phi^b_\nu(-\vec{q})\rangle\sim\frac{\delta_{ab}}{q^2}Q_{\mu\nu},$$

$$Q_{\mu\nu}=\delta_{\mu\nu}-\frac{q_\mu q_\nu}{\mathbf{q}^2},\quad \mu,\nu\neq\tau,\quad \mathbf{q}^2=q_x^2+q_y^2+q_z^2,$$

$$Q_{\mu\nu}=0,\quad \mu\ \text{or}\ \nu=\tau. \quad (19)$$

Using the correlation function Eq. (19), the RG equation at the critical point reads

$$\frac{du}{d\ln l}=\epsilon u-8A(8+N)u^2-4N(A+B)v^2-4(A+B)w^2-4B(8+4N)uv-8(A+B)vw-48Buw,$$

$$\frac{dv}{d\ln l}=\epsilon v-16B(4+N)u^2-4(AN+3BN+4C)v^2-4Cw^2-16(2A+AN+BN+2B)uv-8(A+5B)vw-16(A+3B)uw,$$

$$\frac{dw}{d\ln l}=\epsilon w-64Bu^2-16Bv^2-4(A+6B+BN+2C+CN)w^2-16(B+2C)vw-32Auw,$$

$$\frac{dr}{d\ln l}=2r-8(2A+AN+4B+2BN)ur-8(AN+2BN)vr-8(A+2B)wr. \quad (20)$$

Now $A=8/15$, $B=1/15$, and $C=2/5$, which is different from the isotropic case, due to the different form of the correlation functions. Solving this equation at $r=0$, we find a stable fixed point for large enough N. Again we will not try to write down the lengthy solution of the RG equation, instead we will expand the solution to the order of ϵ/N^2, now the stable fixed point is located at

$$u_*=\frac{27\epsilon}{112N}-\frac{1431\epsilon}{490N^2}+O\left(\frac{\epsilon}{N^3}\right),$$

$$v_*=-\frac{3\epsilon}{56N}-\frac{879\epsilon}{490N^2}+O\left(\frac{\epsilon}{N^3}\right),$$

$$w_* = \frac{15\epsilon}{28N} - \frac{1983\epsilon}{490N^2} + O\left(\frac{\epsilon}{N^3}\right). \tag{21}$$

Again, since $v_*+w_*=2u_*$, this fixed point has an enlarged $O(N)\times O(d-1)$ symmetry. Close to the stable fixed point, the three eigenvectors of the RG flow have scaling dimensions

$$\Delta_1 = -\epsilon + O\left(\frac{\epsilon}{N^2}\right),$$

$$\Delta_2 = -\epsilon + \frac{942\epsilon}{25N} + O\left(\frac{\epsilon}{N^2}\right),$$

$$\Delta_3 = -\epsilon + \frac{1104\epsilon}{175N} + O\left(\frac{\epsilon}{N^2}\right). \tag{22}$$

Δ_2 is the largest scaling dimension. The critical N is $N_c \sim 59$, below which the stable fixed point disappears. Plug the fixed point value Eq. (21) back to the last RG equation in Eq. (20), we obtain the scaling dimension of r at the fixed point

$$[r] = \frac{1}{\nu} = 2 - \epsilon + \frac{138\epsilon}{7N} + O\left(\frac{\epsilon}{N^2}\right). \tag{23}$$

If we keep lowering the temperature after the order of ϕ_x^a and ϕ_y^a then eventually ϕ_z^a will also order at temperature $T_{c2}<T_{c1}$. After the order of ϕ_x^a and ϕ_y^a, the $O(N)$ symmetry is broken down to its subgroup. Let us first assume $w<0$ in $\mathcal{F}_4$, i.e., the three flavors of $O(N)$ vectors are collinear with each other in the low-temperature ordered phase. Let us assume that in the intermediate phase the expectation value $\langle\phi_x^a\rangle \sim \langle\phi_y^a\rangle \sim (1,0,\ldots,0)$, now the $O(N)$ symmetry is broken down to $O(N-1)$ symmetry generated by Lie algebra Γ_{ab} with $a,b=2,\ldots,N$. This symmetry breaking implies that the degeneracy of the N components of ϕ_z^a will be lifted at T_{c2}. Because $w<0$, the nonzero expectation value of ϕ_x^1 and ϕ_y^1 will prefer ϕ_z^1 to order next at lower temperature, which is essentially an Ising transition with order parameter ϕ_z^1. From now on we will denote ϕ_z^1 as ϕ_z. ϕ_z will couple to the gapped fluctuations $\phi_x=\phi_x^1-\langle\phi_x^1\rangle$ and $\phi_y=\phi_y^1-\langle\phi_y^1\rangle$ through the constraint, and the entire low-energy field theory at T_{c2} reads

$$\mathcal{F} = \phi_z(-\nabla^2)\phi_z + \sum_{\mu=x,y}\phi_\mu(-\nabla^2+m^2)\phi_\mu + g\left(\sum_\mu \nabla_\mu\phi_\mu\right)^2 + O(\phi_z^4). \tag{24}$$

After taking the limit $g\to\infty$, the critical correlation function of ϕ_z reads

$$\lim_{g\to\infty} D_{zz}(\vec{q}) \sim \frac{1}{m^2\dfrac{q_z^2}{q_x^2+q_y^2} + q_x^2 + q_y^2 + \cdots}. \tag{25}$$

Therefore this transition at T_{c2} is effectively a $z=2$ transition with scaling dimension $[q_z]=2[q_x]=2[q_y]=2$. Now the total effective dimension is 4, and the $(\phi_z)^4$ term is a marginally irrelevant perturbation, therefore this transition is a mean-field transition with logarithmic corrections. Notice that the Goldstone modes after the $O(N)$ to $O(N-1)$ symmetry breaking are harmless to this transition. The Goldstone mode can be described by $\delta\phi_x^a \sim \delta\phi_y^a \sim (0,\pi^2,\ldots,\pi^N)$ that forms a vector representation of $O(N-1)$, and in order to guarantee the gaplessness of the Goldstone modes, close to T_{c2} the following coupling between ϕ_z and $\vec{\pi}$ is the lowest-order coupling that is allowed:

$$\mathcal{F}_{goldstone} \sim \sum_\mu \phi_z^2(\nabla_\mu\vec{\pi})^2. \tag{26}$$

This term only generates irrelevant perturbations for ϕ_z at the transition. Couplings like $\phi_z^2(\vec{\pi})^2$ is forbidden due to its ability to renormalize the mass of Goldstone mode $\vec{\pi}$.

If $w>0$ in $\mathcal{F}_4$, then the three flavors of $O(N)$ vectors are perpendicular to each other in the low-temperature ordered phase. Let us assume that in the intermediate phase between T_{c1} and T_{c2} the expectation values $\langle\phi_x^a\rangle \sim (1,0,\ldots,0)$ and $\langle\phi_y^a\rangle \sim (0,1,0,\ldots,0)$. Now the $O(N)$ symmetry is broken down to the $O(N-2)$ symmetry generated by Γ_{ab} with $a,b=3,\ldots,N$. Due to the presence of w term in $\mathcal{F}_4$, at T_{c2} the order parameters should be ϕ_z^a with $a=3,\ldots,N$, which forms a vector representation of $O(N-2)$. In the intermediate phase there are in total $2N-3$ Goldstone modes, they are $\pi_x^a=(0,0,\pi_x^3,\ldots,\pi_x^N)$ that corresponds to the $O(N)$ Lie algebra elements Γ_{1a} with $a=3,\ldots,N$; $\pi_y^a=(0,0,\pi_y^3,\ldots,\pi_y^N)$ that correspond to the $O(N)$ Lie algebra elements Γ_{2a} with $a=3,\ldots,N$; plus a rotation mode between ϕ_x^1 and ϕ_x^2, denoted as Γ_{12}. Again the order parameter ϕ_z^a will couple to these Goldstone modes through the constraint. The field theory close to T_{c2} reads

$$\mathcal{F} = \sum_{\mu=x,y}\sum_{a=3}^{N} K\pi_\mu^a(-\nabla^2)\pi_\mu^a + \sum_{a=3}^{N}\phi_z^a(-\nabla^2+r)\phi_z^a + \sum_{a=3}^{N} g(\nabla_z\phi_z^a + \nabla_x\pi_x^a + \nabla_y\pi_y^a)^2 + O[(\phi_z^a)^4]. \tag{27}$$

K is in general not one and will flow under RG. But at the first order of ϵ expansion it can be taken as a constant. Notice that the interaction between Goldstone modes π_μ^a have to be irrelevant to guarantee their gaplessness. The universality class of this transition at $r=0$ can be calculated using the correlation function of ϕ_z^a after taking the limit $g\to\infty$

$$\lim_{g\to\infty, r\to 0} D_{zz}^{ab} = \frac{P_{zz}}{q_x^2+q_y^2+Kq_z^2}\delta_{ab}. \tag{28}$$

$P_{\mu\nu}$ is the transverse projection matrix. The first-order ϵ expansion leads to the following scaling dimension and critical exponent:

$$[r] = \frac{1}{\nu} = 2 - \frac{N-1}{N+5}\epsilon, \tag{29}$$

which is identical to the 3D $O(N-2)$ Wilson-Fisher (WF) fixed point although higher-order expansions may deviate from the WF fixed point. Therefore our formalism implies that with large enough N, the easy-plane anisotropy will split the transition discussed in the previous section to two second-order transitions.

IV. $O(N)$ SPIN ICE WITH EASY AXIS ANISOTROPY

In model Eq. (17), if we make the following choice of parameters: $J_{1,z}>J_{1,x}=J_{1,y}>0$ and $J_{2,zx}=J_{2,zy}>J_{2,xy}=J_{2,xz}=J_{2,yx}=J_{2,yz}$, then when we lower the temperature from the algebraic phase, the $O(N)$ vectors along the z axis will order first at temperature T_{c1}. Similar to the previous section, this easy-axis anisotropy can be described by an extra mass gap m for both ϕ_x^a and ϕ_y^b modes in the field theory Eq. (8), and ϕ_z^a becomes the only order parameter at low energy. However, now we can no longer take the limit $m\to\infty$ because with this limit all the correlation functions will vanish. If we keep m finite, the correlation function of ϕ_z^a takes a similar form as Eq. (25)

$$\lim_{g\to\infty} D_{zz}^{ab}(\vec{q}) \sim \frac{\delta_{ab}}{m^2\frac{q_z^2}{q_x^2+q_y^2}+q_x^2+q_y^2+\cdots} \tag{30}$$

therefore this transition is again an effective a $z=2$ mean-field transition. A similar situation with easy-axis anisotropy has been studied in Ref. 20 although there the easy axis was along the diagonal direction.

If the temperature is lowered even more from T_{c1} then at T_{c2} the spin vectors within xy planes will order next. Again the nature of this transition would depend on the sign of w in $\mathcal{F}_4$. If $w<0$, i.e., the three flavors of $O(N)$ vectors order collinearly then if ϕ_z^a orders along direction $\langle\phi_z^a\rangle \sim(1,0,\ldots,0)$, at temperature T_{c2} ϕ_x^1 and ϕ_y^1 will become the critical modes. ϕ_x^1 and ϕ_y^1 are coupled to the gapped fluctuation $\phi_z=\phi_z^1-\langle\phi_z^1\rangle$, and the low-energy field theory describing ϕ_x and ϕ_y is identical to Eq. (18) with $N=1$. By solving the RG equation Eq. (20) with $N=1$, no stable fixed point is found.

If $w>0$, the critical modes at T_{c2} is ϕ_x^a and ϕ_y^a with $a=2,\ldots,N$, which form vector representations of $O(N-1)$. These two critical modes are coupled to gapless Goldstone mode $\pi_z^a=(0,\pi_z^2,\ldots,\pi_z^N)$ with field theory

$$\mathcal{F}=\sum_{\mu=x,y}\sum_{a=2}^{N}\phi_\mu^a(-\nabla^2+r)\phi_\mu^a+\sum_{a=2}^{N}K\pi_z^a(-\nabla^2)\pi_z^a + \sum_{a=3}^{N} g(\nabla_z\pi_z^a+\nabla_x\phi_x^a+\nabla_y\phi_y^a)^2+\mathcal{F}_4(\phi_x^a,\phi_y^a). \tag{31}$$

Now $\mathcal{F}_4$ only involves two flavors of $O(N-1)$ vectors ϕ_x^a and ϕ_y^a. The RG equation for $\mathcal{F}_4$ can be calculated in the same manner as we did before. If we take $K=1$, then the RG equation for $\mathcal{F}_4$ takes the same form as Eq. (20) but with $A=5/8$, $B=1/24$, and $C=13/24$. With large enough N there is still a stable fixed point and the answer is qualitatively unchanged when K deviates from one slightly.

V. SUMMARY AND DISCUSSION

In this work we studied the transition between high-temperature Coulomb phase and the low-temperature staggered spin ordered phase in the $O(N)$ spin-ice model. Higher-order ϵ expansion is demanded to obtain more quantitatively accurate results. The model Eq. (5) can be simulated directly numerically, and a comparison between the critical exponents obtained from a higher-order RG calculation and the simulation can be used as a test of our constrained GL field theory Eq. (8). The columnar phase with $J_1>0$ and $J_2<0$ is also interesting. But since the columnar order does not correspond to the uniform order of ϕ_μ^a, the order-parameter description is more complicated. The columnar order is equivalent to order of ϕ_x^a at momentum $Q_1=(0,\pi,\pi)$, ϕ_y^a at momentum $Q_1=(\pi,0,\pi)$, and ϕ_z^a at momentum $Q_1=(\pi,\pi,0)$. Therefore presumably we could describe this transition with condensation of ϕ_μ^a at all three wave vectors. However, just like the Coulomb-columnar transition in CDM-1 discussed in Sec. I, some topological configuration of these order parameters may be forbidden, which potentially can change the universality class completely. We will study the phase transition of the columnar order in future.

The CDM is also considered as a simple analog of the spin-ice materials such as $Ho_2Ti_2O_7$ and $Dy_2Ti_2O_7$,[21–23] where the Ho^{3+} and Dy^{3+} magnetic moments reside on the sites of a pyrochlore lattice, and the ground state of these moments satisfies the same ice-rule constraint as Eq. (4). The Coulomb phase of the spin ice materials with fractionalized "monopole" like defect excitation has been observed experimentally.[24] The formalism developed in our work is largely applicable to the pyrochlore lattice while the symmetry analysis and the number of quartic terms are different. A complete symmetry analysis is demanded in order to correctly understand the $O(N)$ spin ice on the pyrochlore lattice.

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[1] D. A. Huse, W. Krauth, R. Moessner, and S. L. Sondhi, Phys. Rev. Lett. **91**, 167004 (2003).
[2] G. Chen, J. Gukelberger, S. Trebst, F. Alet, and L. Balents, Phys. Rev. B **80**, 045112 (2009).
[3] F. Alet, G. Misguich, V. Pasquier, R. Moessner, and J. L. Jacobsen, Phys. Rev. Lett. **97**, 030403 (2006).
[4] G. Misguich, V. Pasquier, and F. Alet, Phys. Rev. B **78**, 100402(R) (2008).
[5] R. Moessner and S. L. Sondhi, Phys. Rev. B **68**, 184512 (2003).
[6] M. Hermele, T. Senthil, and M. P. A. Fisher, Phys. Rev. B **72**, 104404 (2005).
[7] O. I. Motrunich and T. Senthil, Phys. Rev. B **71**, 125102 (2005).
[8] S. Powell and J. T. Chalker, Phys. Rev. Lett. **101**, 155702 (2008).

[9] O. I. Motrunich and A. Vishwanath, Phys. Rev. B **70**, 075104 (2004).
[10] M. Kamal and G. Murthy, Phys. Rev. Lett. **71**, 1911 (1993).
[11] O. Motrunich and A. Vishwanath, arXiv:0805.1494 (unpublished).
[12] L. Pauling, J. Am. Chem. Soc. **57**, 2680 (1935).
[13] C. Wu, J. P. Hu, and S. C. Zhang, Phys. Rev. Lett. **91**, 186402 (2003).
[14] A. Gorshkov, M. Hermele, V. Gurarie, C. Xu, P. Julienne, J. Ye, P. Zoller, E. Demler, M. Lukin, and A. Rey, Nat. Phys. **6**, 289 (2010).
[15] C. L. Henley, Phys. Rev. Lett. **62**, 2056 (1989).
[16] A. Aharony, Phys. Rev. B **12**, 1049 (1975).
[17] The author is grateful to Leon Balents for pointing this out.
[18] M. E. Fisher and A. Aharony, Phys. Rev. Lett. **30**, 559 (1973).
[19] A. Aharony and M. E. Fisher, Phys. Rev. B **8**, 3323 (1973).
[20] T. S. Pickles, T. E. Saunders, and J. T. Chalker, EPL **84**, 36002 (2008).
[21] P. W. Anderson, Phys. Rev. **102**, 1008 (1956).
[22] S. V. Isakov, R. Moessner, and S. L. Sondhi, Phys. Rev. Lett. **95**, 217201 (2005).
[23] C. Castelnovo, R. Moessner, and S. L. Sondhi, Nature (London) **451**, 42 (2008).
[24] T. Fennell, P. P. Deen, A. R. Wildes, K. Schmalzl, D. Prabhakaran, A. T. Boothroyd, R. J. Aldus, D. F. McMorrow, and S. T. Bramwell, Science **326**, 415 (2009).